Abstract

In 2001, A.V. Borisov and I.S. Mamaev discovered a new integrable case on the Lie algebras $e(3)$, $so(4)$ and $so(3,1)$. This is a Hamiltonian system with two degrees of freedom, where both the Hamiltonian and additional integral are homogeneous polynomials of degrees 2 and 4, respectively. In this paper, bifurcation diagram of the Hamiltonian for the integrable case under consideration on the Lie algebras $e(3)$, $so(4)$ and $so(3,1)$ for different values of parameters is constructed.

Keywords: Integrable Hamiltonian systems, critical points, bifurcation diagram.

1 Introduction

The foundations of the theory of topological classification of integrable Hamiltonian systems with two degrees of freedom was established in [4,5]. In the sequel, various methods of calculation of invariants classifying systems on isoenergy surfaces were developed in [2]. Those invariants were calculated for many classic integrable cases appearing in the mechanics and mathematical physics. We construct the bifurcation diagram of one of integrable cases revealed recently (Borisov-Mamaev case on $e(3)$, $so(4)$ and $so(3,1)$).

1.1 Euler’s equations on Lie algebras

Let $g$ be a finite-dimensional Lie algebra and $g^*$ the corresponding coalgebra (the space of linear functions on $g$). Consider a basis $e_1, e_2, ..., e_n$ in the Lie algebra $g$ and the corresponding structural constants $c_{ij}^k$ of the algebra $g$ in this basis. Thus, the Lie bracket corresponding to the Lie algebra $g$ is written in the following form:

$$[e_i, e_j] = c_{ij}^k e_k$$

Let $x_1, x_2, ..., x_n$ be the linear coordinates on $g^*$ corresponding to the basis $e_1, e_2, ..., e_n$.

Definition 1.1 The Poisson bracket on $g^*$ defined by the formula

$$\{f, g\}(x) = c_{ij}^k x_k \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$$

where $f, g$ are smooth functions on $g^*$, is called the Lie–Poisson bracket for the Lie algebra $g$.

Definition 1.2 The equations

$$\dot{x}_i = \{x_i, H\}$$

which define a dynamical system on $g^*$, where $H$ is a smooth function on $g^*$, are called Euler’s equations for the Lie algebra $g$. 

It is well known (see, for example, [7]) that the dynamical system defined by Euler’s equations is Hamiltonian (with Hamiltonian $H$) on orbits of the coadjoint representation of the Lie algebra $g$. The corresponding vector field on the orbits is called the skew-gradient of the function $H$ and is denoted by $sgradH$.

Many dynamical systems describing mechanical and physical problems can be written in the form of Euler’s equations for a certain Lie algebra. For example, various problems on the motion of a rigid body are described by Euler’s equations for the Lie algebra $e(3)$ (see [3,6,7]). Those systems are Hamiltonian systems with two degrees of freedom on orbits of the Lie algebra $e(3)$. The integrability of such systems means the existence of an integral that is functionally independent with the Hamiltonian $H$ on orbits. In the section 1.2, we describe an integrable system on the Lie algebras $e(3)$, $so(4)$ and $so(3,1)$ with quadratic Hamiltonian and integral of degree of four, which was found by A.V. Borisov and I.S. Mamaev in [1].

1.2 Borisov-Mamaev Integrable case on the Lie Algebras $e(3)$, $so(4)$ and $so(3,1)$

On the space $e(3)^*$ dual to the Lie algebra $e(3)$ we consider linear coordinates $S_1, S_2, S_3, R_1, R_2, R_3$ for which the Lie-Poisson bracket has the form

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k, \{S_i, R_j\} = \varepsilon_{ijk} R_k, \{R_i, R_j\} = 0$$

Also on the space $so(4)^*$ dual to the Lie algebra $so(4)$ Lie-Poisson bracket has the form

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k, \{S_i, R_j\} = \varepsilon_{ijk} R_k, \{R_i, R_j\} = \varepsilon_{ijk} S_k$$

And also on the space $so(3,1)^*$ dual to the Lie algebra $so(3,1)$ Lie-Poisson bracket has the form

$$\{S_i, S_j\} = \varepsilon_{ijk} S_k, \{S_i, R_j\} = \varepsilon_{ijk} R_k, \{R_i, R_j\} = -\varepsilon_{ijk} S_k$$

where $\varepsilon_{ijk}$ is the sign of the permutation $(123) \rightarrow (ijk)$. It is convenient to regard the coordinates $(S_1, S_2, S_3)$ and $(R_1, R_2, R_3)$ as components of two three-dimensional vectors $S$ and $R$.

Euler’s equations on $e(3)^*$, $so(4)^*$ and $so(3,1)^*$ with Hamiltonian $H$ take the following form:

$$\dot{S}_i = \{S_i, H\}, \dot{R}_i = \{R_i, H\}$$

The Euler equations mentioned above always have two Casimir functions on the space $e(3)^*$ has the form:

$$f_1 = R^2, f_2 = \langle S, R \rangle$$

Also on the space $so(4)^*$ has the form:

$$f_1 = S^2 + R^2, f_2 = \langle S, R \rangle$$

And also for the space $so(3,1)^*$ has the form:

$$f_1 = -S^2 + R^2, f_2 = \langle S, R \rangle$$

where $\langle ., . \rangle$ is the Euclidean scalar product in $\mathbb{R}^3$ (in particular, $S^2$ and $R^2$ denote the scalar squares of the vectors $S$ and $R$). The functions $f_1$ and $f_2$ commute with respect to the Lie-Poisson bracket with all functions, and their common levels

$$M_{c,g} = \{(S, R)| f_1(S, R) = c, f_2(S, R) = g\}$$

are orbits of the coadjoint representation of the Lie algebra $g = e(3), so(4), so(3,1)$. The restriction of the Lie-Poisson bracket to $M_{c,g}$ is nondegenerate, that is, it defines a symplectic structure on orbits. The Hamiltonian $H$ and additional integral $K$ in Borisov-Mamaev case on the Lie algebra $e(3)$ have the following form:
2. Critical \( R \) and arbitrary \( c \) is reduced to the case \( c = 1 \). So in what follows we shall consider only orbits \( M_{1,g}^4 \). Moreover, without loss generality, one can assume that \( 0 \leq g \) since, for example, the coordinate transformation

\[
(S_1, S_2, S_3, R_1, R_2, R_3) \rightarrow (-S_1, S_2, S_3, R_1, -R_2, -R_3)
\]

preserves the invariant \( f_1 \), the Hamiltonian \( H \), and the integral \( K \) and changes the sign of the invariant \( f_2 \). Similarly, one can decrease the number of cases for the parameter \( \alpha \) to be considered. The transformation \( S_i \rightarrow -S_i \), \( R_i \rightarrow -R_i \) and \( \alpha \rightarrow -\alpha \) preserve the invariants \( f_1 \), \( f_2 \) and the integral \( K \) while the Hamiltonian \( H \) merely changes the sign (this does not affect the topology of the system under consideration), thus one can assume that \( \alpha > 0 \).

2 The bifurcation diagram for the Hamiltonian \( H \)

By definition, bifurcation values of the Hamiltonian are critical values of the map \( H : M_{1,g}^4 \rightarrow \mathbb{P}(h) \). Critical points of the Hamiltonian are zeros of its skew gradient.

We can write out the vector field \( sgrad \ H \), for the Hamiltonian \( H \) on the Lie algebra \( e(3) \) explicitly, in the following form:

\[
\begin{align*}
\{S_1, H\} &= 2\alpha S_2 S_3 + S_i R_j - S_j R_i, \\
\{S_2, H\} &= S_i R_j - S_j R_i, \\
\{S_3, H\} &= -2\alpha S_2 S_3, \\
\{R_1, H\} &= 4\alpha S_2 R_3 - 2\alpha S_3 R_2 - R_i R_j, \\
\{R_2, H\} &= -2\alpha S_2 R_3 + 2\alpha S_3 R_2 - R_i R_j, \\
\{R_3, H\} &= 2\alpha S_2 R_3 - 4\alpha S_2 R_1 + R_i^2 + R_j^2.
\end{align*}
\]

setting \( sgrad \ H \) equal to zero we obtain the following results.

Theorem 2.1 The critical points of the Hamiltonian \( H \) are formed by the following two-parameter families (the
skew gradient  $\text{grad}\, K$ also vanishes at these point) in the space  $e(3) = P^6(S, R):$

1) $(0, 0, S_j, 0, 0, R_j)\) 

2) $(0, S_2, 0, R_1, R_2, 0); \text{ where } -4\alpha S_2 R_1 + R_1^2 + R_2^2 = 0$

3) $(0, S_2, S_3, 2\alpha S_2, 2\alpha S_2, 2\alpha S_3);$ 

4) $(0, S_2, S_3, 2\alpha S_2, -2\alpha S_2, -2\alpha S_3);$ 

5) $(S_1, 0, 0, R_1, R_2, 0); \text{ where } 2\alpha S_1 R_2 + R_1^2 + R_2^2 = 0$

We shall be investigating the system on $M^4_{1,g}$, so we need the additional conditions $f_1 = 1$ and $f_2 = g$.

**Theorem 2.2** The critical points of the Hamiltonian $H$ on $M^4_{1,g}$ are listed below as five series corresponding to the five families from Theorem 2.1:

1) For any $g$, there exist four points of the form 

   $$(0, 0, S_1, 0, 0, R_1); \text{ where } R_1^2 = 1, S_1 R_1 = g$$

   here $h = \alpha g^2$.

2) For any $g$, there exist four points of the form 

   $$(0, S_2, 0, \frac{1}{4\alpha S_2}, \frac{g}{S_2}, 0); \text{ where } S_2^2 = \frac{16\alpha^2 g^2 + 1}{16\alpha^2}$$

   here $h = \frac{16\alpha^2 g^2 - 1}{8\alpha}$.

3) For any $\frac{1}{4\alpha} \leq g \leq \frac{1}{2\alpha}$ there exist four points of the form 

   $$(0, S_2, S_3, 2\alpha S_2, 2\alpha S_2, 2\alpha S_3)$$

   where 

   $$S_2^2 = \frac{1 - 2\alpha g}{4\alpha^2}; \quad S_3^2 = \frac{4\alpha g - 1}{4\alpha^2}$$

   here $h = \frac{4\alpha g - 1}{4\alpha}$.

4) For any $\frac{1}{2\alpha} \leq g \leq \frac{1}{4\alpha}$ there exist four points of the form 

   $$(0, S_2, S_3, 2\alpha S_2, -2\alpha S_2, -2\alpha S_3)$$

   where 

   $$S_2^2 = \frac{1 + 2\alpha g}{4\alpha^2}; \quad S_3^2 = -\frac{4\alpha g + 1}{4\alpha^2}$$

   here $h = -\frac{4\alpha g + 1}{4\alpha}$.

5) For any $g$ there exist four points of the form 

   $$(S_1, 0, 0, \frac{g}{S_1}, \frac{1}{2\alpha S_1}, 0)$$

   where
\[
S^2_i = \frac{1 + 4\alpha^2 g^2}{4\alpha^2}
\]

here \( h = \frac{4\alpha^2 g^2 - 1}{4\alpha} \).

**Theorem 2.3** The bifurcation diagram on the case \( e(3) \) in the plane \( P^2(g, h) \) for the mapping \( f_z \times H \) consists of the following one-dimensional trajectories:

1) \( h = ag^2; \ g \in P \)
2) \( h = 2ag^2 - \frac{1}{8\alpha}; \ g \in P \)
3) \( h = ag^2 - \frac{1}{4\alpha}; \ g \in P \)
4) \( h = g - \frac{1}{4\alpha}; \ \frac{1}{4\alpha} \leq g \leq \frac{1}{2\alpha} \)
5) \( h = -g - \frac{1}{4\alpha}; -\frac{1}{2\alpha} \leq g \leq -\frac{1}{4\alpha} \)

Note that the bifurcation diagram is symmetric about the axis \( g = 0 \).

![Figure 1: Bifurcation diagram for Hamiltonian \( H \) on the Lie algebra \( e(3) \)](image)

Also we can write out the vector field \( \text{grad} \ H \), for the Hamiltonian \( H \) on the Lie algebra \( so(4) \) explicitly, in the following form:

\[
\{S_1, H\} = 2\alpha S_2 S_3 + S_2 R_3 - S_3 R_1,
\]

\[
\{S_2, H\} = \frac{1}{2\alpha} S_1 S_3 + S_2 R_3 - S_3 R_2,
\]

\[
\{S_3, H\} = -2(\alpha + \frac{1}{4\alpha}) S_2 S_2,
\]

\[
\{R_1, H\} = 4\alpha S_2 R_3 - 2\alpha S_2 R_2 + S_3 S_3 - R_3 R_3,
\]

\[
\{R_2, H\} = -2(\alpha - \frac{1}{4\alpha}) S_1 R_3 + 2\alpha S_3 R_1 - R_2 R_3 + S_2 S_3,
\]
\[ \{ R_3, H \} = 2(\alpha - \frac{1}{4\alpha})S_1R_2 - 4\alpha S_2 R_1 + R_1^2 + R_2^2 - S_1^2 - S_2^2 . \]

setting \( sgrad \ H \) equal to zero we obtain the following results.

**Theorem 2.4** The critical points of the Hamiltonian \( H \) are formed by the following two-parameter families (the skew gradient \( sgrad \ K \) also vanishes at these point) in the space \( so(4) = D^4(S, R) \):

1) \((0,0, S_1, 0,0, R_1)\)
2) \((0, S_2, 0, R_1, R_2, 0)\); where \(-4\alpha S_2 R_1 + R_1^2 + R_2^2 - S_2^2 = 0\)
3) \((0, S_2, S_3, 2\alpha S_2, \sqrt{4\alpha^2 + 1} S_2, \sqrt{4\alpha^2 + 1} S_2)\);
4) \((0, S_2, S_3, 2\alpha S_2, -\sqrt{4\alpha^2 + 1} S_2, -\sqrt{4\alpha^2 + 1} S_2)\);
5) \((S_1, 0,0, R_1, R_2, 0)\); where \(2(\alpha - \frac{1}{4\alpha})S_1R_2 + R_1^2 + R_2^2 - S_1^2 = 0\)

**Theorem 2.5** The critical points of the Hamiltonian \( H \) on \( M_{4,g} \) are listed below as five series corresponding to the five families from Theorem 2.4:

1) For any \( 0 \leq g \leq \frac{1}{2} \), there exist four points of the form

\[
(0,0, S_1, 0,0, R_1); \text{where} S_3^2 + R_3^2 = 1, S_3 R_3 = g
\]

here \( h = \frac{\alpha}{2} \left(1 + \sqrt{1 - 4g^2}\right) \).

2) For any \( 0 \leq g \leq \frac{1}{2} \), there exist four points of the form

\[
(0, S_2, 0, \frac{1}{4\alpha S_2} - \frac{S_2}{2\alpha}, g, 0); \text{where} S_3^2 = \frac{1}{2} \pm \alpha \sqrt{\frac{1 - 4g^2}{1 + 4\alpha^2}}
\]

here \( h = \alpha \pm \frac{1}{2} \sqrt{(1 + 4\alpha^2)(1 - 4g^2)} \).

3) For any \( \frac{1}{2\sqrt{1 + 4\alpha^2}} \leq g \leq \frac{\sqrt{1 + 4\alpha^2}}{2 + 4\alpha^2} \) there exist four points of the form

\[
(0, S_2, S_3, 2\alpha S_2, \sqrt{1 + 4\alpha^2} S_2, \sqrt{1 + 4\alpha^2} S_2)
\]

where

\[
S_3^2 = \frac{\sqrt{1 + 4\alpha^2} - (2 + 4\alpha^2)g}{4\alpha^2 \sqrt{1 + 4\alpha^2}}; \quad S_3^2 = \frac{2\sqrt{1 + 4\alpha^2} g - 1}{4\alpha^2}
\]

here \( h = \frac{2\sqrt{1 + 4\alpha^2} g - 1}{4\alpha} \).

4) For any \( \frac{\sqrt{1 + 4\alpha^2}}{2 + 4\alpha^2} \leq g \leq -\frac{1}{2\sqrt{1 + 4\alpha^2}} \) there exist four points of the form

\[
(0, S_2, S_3, 2\alpha S_2, -\sqrt{1 + 4\alpha^2} S_2, -\sqrt{1 + 4\alpha^2} S_2)
\]

where
\[ S_2^2 = \frac{\sqrt{1+4\alpha^2} + (2+4\alpha^2)g}{4\alpha^2 \sqrt{1+4\alpha^2}}; S_3^2 = \frac{2\sqrt{1+4\alpha^2}g + 1}{-4\alpha^2} \]

here \( h = -\frac{2\sqrt{1+4\alpha^2}g + 1}{4\alpha} \).

5) For any \( 0 \leq g \leq \frac{1}{2} \) there exist four points of the form

\[ (S_1,0,0, \frac{4\alpha S_1}{4\alpha^2 - 1} - \frac{2\alpha}{(4\alpha^2 - 1)S_1},0) \]

where

\[ S_1^2 = \frac{1}{2} \mp \frac{1}{2} \sqrt{\frac{(1-4g^2)(4\alpha^2 - 1)^2}{(1+4\alpha^2)^3}} \]

here \( h = \frac{4\alpha^2 - 1}{8\alpha} \pm \frac{1}{8\alpha} \sqrt{(1-4g^2)(1+4\alpha^2)^2} \).

**Theorem 2.6** The bifurcation diagram on the case \( so(4) \) in the plane \( P^2(g,h) \) for the mapping \( f_2 \times H \) consists of the following one-dimensional trajectories:

1) \( \left( \frac{2h}{\alpha} - 1 \right)^2 + 4\alpha g^2 = 1 ; \ g \in P \)

2) \( \left( \frac{2h - 2\alpha}{1 + 4\alpha^2} \right) + 4g^2 = 1 ; \ g \in P \)

3) \( \left( \frac{8\alpha h - 4\alpha^2 + 1}{1 + 4\alpha^2} \right)^2 + 4g^2 = 1 ; \ g \in P \)

4) \( h = \frac{\sqrt{1+4\alpha^2}}{2\alpha} - \frac{1}{4\alpha} \leq g \leq \frac{\sqrt{1+4\alpha^2}}{2 + 4\alpha^2} \)

5) \( h = -\frac{\sqrt{1+4\alpha^2} + 1}{2\alpha} - \frac{1}{4\alpha} \leq g \leq -\frac{\sqrt{1+4\alpha^2}}{2 + 4\alpha^2} \)

6) \( \frac{4\alpha^2 - 1}{8\alpha} \leq h \leq \alpha ; \ g = \frac{1}{2} \)

7) \( \frac{4\alpha^2 - 1}{8\alpha} \leq h \leq \alpha ; \ g = -\frac{1}{2} \)

Note that the bifurcation diagram is symmetric about the axis \( g = 0 \).
Figure 2: Bifurcation diagram for Hamiltonian $H$ on the Lie algebra $so(4)$

And also We can write out the vector field $sgrad\ H$, for the Hamiltonian $H$ on the Lie algebra $so(3,1)$ explicitly, in the following form:

$$\{S_1, H\} = 2\alpha S_2 S_3 + S_1 R_3 - S_3 R_1,$$

$$\{S_2, H\} = -\frac{1}{2\alpha} S_1 S_3 + S_2 R_3 - S_3 R_2,$$

$$\{S_3, H\} = -2(\alpha - \frac{1}{4\alpha}) S_1 S_2,$$

$$\{R_1, H\} = 4\alpha S_2 R_3 - 2\alpha S_3 R_2 - S_1 S_3 - R_1 R_3,$$

$$\{R_2, H\} = -2(\alpha + \frac{1}{4\alpha}) S_1 R_3 + 2\alpha S_3 R_1 - R_1 R_3 - S_2 S_3,$$

$$\{R_3, H\} = 2(\alpha + \frac{1}{4\alpha}) S_1 R_2 - 4\alpha S_2 R_1 + R_1^2 + R_2^2 + S_1^2 + S_2^2.$$

setting $sgrad\ H$ equal to zero we obtain the following results.

**Theorem 2.7** For an arbitrary $\alpha \neq \frac{1}{2}$ the critical points of the Hamiltonian $H$ are formed by the following two-parameter families (the skew gradient $sgrad\ K$ also vanishes at these point) in the space $so(3,1)^* = P^6(S,R)$:

1)$(0,0,S_3,0,0,R_3)$

2)$(0,S_2,0,R_1,R_2,0)$; where $-4\alpha S_2 R_1 + R_1^2 + R_2^2 + S_1^2 = 0$
3) \((0, S_2, S_3, 2\alpha S_2, \sqrt{4\alpha^2 - 1} S_2, \sqrt{4\alpha^2 - 1} S_3); \) for \(\alpha > \frac{1}{2}\)

4) \((0, S_2, S_3, 2\alpha S_2, -\sqrt{4\alpha^2 - 1} S_2, -\sqrt{4\alpha^2 - 1} S_3); \) for \(\alpha > \frac{1}{2}\)

5) \((S_1, 0, 0, R_1, R_2, 0); \) where \(2(\alpha + \frac{1}{4\alpha}) S_1 R_2 + R_1^2 + R_2^2 + S_1^2 = 0\)

**Theorem 2.8** For any \(\alpha > \frac{1}{2}\) the critical points of the Hamiltonian \(H\) on \(M_4^4\) are listed below as five series corresponding to the five families from Theorem 2.7:

1) For any \(g\), there exist four points of the form

\[(0, 0, S_1, 0, 0, R_1); \text{where } -S_3^2 + R_3^2 = 1, S_3 R_3 = g\]

where \(h = \frac{\alpha}{2} (-1 + \sqrt{1 + 4g^2})\).

2) For any \(g\), there exist four points of the form

\[(0, S_2, 0, \frac{1}{4\alpha} S_2, S_2, g, 0); \text{where } S_2^2 = -\frac{1}{2} + \alpha \sqrt{\frac{1 + 4g^2}{4\alpha^2 - 1}}\]

where \(h = \alpha + \frac{1}{2} \sqrt{(4\alpha^2 - 1)(1 + 4g^2)}\).

3) For any \(\frac{1}{2\sqrt{4\alpha^2 - 1}} \leq g \leq \frac{\sqrt{4\alpha^2 - 1}}{4\alpha^2 - 2}\) there exist four points of the form

\[(0, S_2, S_3, 2\alpha S_2, \sqrt{4\alpha^2 - 1} S_2, \sqrt{4\alpha^2 - 1} S_3)\]

where

\[S_2^2 = \frac{\sqrt{4\alpha^2 - 1} - (4\alpha^2 - 2)g}{4\alpha^2 \sqrt{4\alpha^2 - 1}}; S_3^2 = \frac{2\sqrt{4\alpha^2 - 1}g - 1}{4\alpha^2}\]

where \(h = \frac{2\sqrt{4\alpha^2 - 1}g - 1}{4\alpha}\).

4) For any \(\frac{\sqrt{4\alpha^2 - 1}}{4\alpha^2 - 2} \leq g \leq -\frac{1}{2\sqrt{4\alpha^2 - 1}}\) there exist four points of the form

\[(0, S_2, S_3, 2\alpha S_2, -\sqrt{4\alpha^2 - 1} S_2, -\sqrt{4\alpha^2 - 1} S_3)\]

where

\[S_2^2 = \frac{\sqrt{4\alpha^2 - 1} + (4\alpha^2 - 2)g}{4\alpha^2 \sqrt{4\alpha^2 - 1}}; S_3^2 = \frac{2\sqrt{4\alpha^2 - 1}g + 1}{-4\alpha^2}\]

where \(h = -\frac{2\sqrt{4\alpha^2 - 1}g + 1}{4\alpha}\).

5) For any \(g\) there exist four points of the form

\[(S_1, 0, 0, \frac{g}{S_1}, \frac{4\alpha S_1}{4\alpha^2 + 1} - \frac{2\alpha}{(4\alpha^2 + 1)S_1}, 0)\]
where

\[ S_1^2 = \frac{-1}{2} + \frac{1}{2} \sqrt{(1+4g^2)(4\alpha^2 +1)^2 \over (4\alpha^2 -1)^2} \]

here \( h = -\frac{4\alpha^2 +1}{8\alpha} \quad \text{and} \quad \frac{1}{8\alpha} \sqrt{(1+4g^2)(4\alpha^2 -1)^2} \).

**Theorem 2.9** The bifurcation diagram in the case \( so(3,1) \) the plane \( P^2(g,h) \) for the mapping \( f_2 \times H \) consists of the following one-dimensional trajectories:

1) \( \left( \frac{2h}{\alpha} -1 \right)^2 - 4\alpha g^2 = 1 \quad g \in P \)

2) \( \frac{(2h + 2\alpha)^2}{4\alpha^2 -1} - 4g^2 = 1 \quad g \in P \)

3) \( \frac{(8\alpha h + 4\alpha^2 +1)(-1)^2}{4\alpha^2} - 4g^2 = 1 \quad g \in P \)

4) \( h = \frac{\sqrt{4\alpha^2 -1}}{2\alpha} g - \frac{1}{4\alpha} \quad \frac{1}{2\sqrt{4\alpha^2 -1}} \leq g \leq \frac{\sqrt{4\alpha^2 -1}}{4\alpha^2 -2} \)

5) \( h = -\sqrt{4\alpha^2 -1} \quad g - \frac{1}{4\alpha} \quad \frac{-\sqrt{4\alpha^2 -1}}{4\alpha^2 -2} \leq g \leq -\frac{1}{2\sqrt{4\alpha^2 -1}} \)

Note that the bifurcation diagram is symmetric about the axis \( g = 0 \).

Figure 3: Bifurcation diagram for Hamiltonian \( H \) on the Lie algebra \( so(3,1) \)

**References**
