Developing a model for Capacitated Hierarchical hub location with considering Delivery Time Restriction

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ABSTRACT

In this paper, a three-level hierarchical p-hub median problem is considered where the complete network at the first level consists of the central hubs. The second and third levels consist of hub and demand nodes respectively, which are connected through star networks. Despite the problems considered in the literature of the hierarchical p-hub median problem, all the nodes and arcs on the network have limited capacities. Besides, delivery time restriction is taken into account. The problem is to decide on the locations of a predetermined number of hubs and central hubs among the available nodes in order to minimize the total costs. The mathematical formulation of the problem is given and the results of solving the problem are given and analyzed.

KEYWORDS: HUB LOCATION; P-HUB MEDIAN; HIERARCHICAL; SINGLE ASSIGNMENT.
1. **Introduction**

In network design, considering a complete network for all the nodes increases the costs drastically. Consider a computer network in which all pairs of computers are directly connected to each other. The space occupied and the wire needed will obviously cause serious difficulties and increase the costs. In hub location problems, some facilities, called hubs, are used in order to consolidate and distribute traffic and avoid the use of direct links. The traffic may consist of data, merchandise, passengers, etc. Three possible functions are considered for a hub:

1. Consolidation in order to make use of the profits of the economies of scale.
2. Switching the traffics to different nodes.
3. Distribution of the traffics.

In general, $p$-hub location problems consist of deciding on the location of hubs and the connections in order to transport the commodities through the network while minimizing the total costs or transportation times.

The $p$-hub location problem consists of determining the locations of the hubs and allocating the demand nodes to these hubs. Considering the allocation process, these problems may be classified into two different groups. The first group is known as single allocation (or single assignment) where each destination node is connected to only one of the hubs in the network. On the other hand, in multiple allocation (or multiple assignment) problems each node may be served by more than one hub. The problem considered in this paper is single allocation, since each one of the demand nodes is connected to only one hub or central hub.


In this paper, the hierarchical $p$-hub median location problem is considered. Despite the existing literature of the problem, capacity and delivery time restrictions are taken into account that will help to resemble the real world situations and thus, makes the solution procedure more difficult.

The rest of the paper is organized as follows. The problem is described in details in Section 2. The mathematical model formulation of the problem is presented in Section 3. Section 4 includes the solution results of the problem and in Section 5 the concluding remarks are presented.

2. **Problem description**

Consider Figure 1 where the set of nodes is given along with the traffic demand between the nodes and transportation cost between pairs of nodes. The problem is to construct a network to route the traffics while minimizing the costs.
A three-level hierarchical $p$-hub median approach is implemented to model the problem. In Figure 1, circles indicate demand nodes which are candidates for hub locations (third level). These nodes are connected with a star network. Hubs, indicated by squares, are connected to each other through a star network as well (second level). Hexagons represent central hubs which are chosen amongst the hub nodes and are connected in a complete network (first level).

In this network, the route for a specific traffic from an origin to a destination may include up to four hubs. If both origin and destination nodes are assigned to the same hub, the traffic is routed through the path origin-hub-destination. If the origin and destination nodes are assigned to two different hubs, hub A and B respectively, which are assigned to the same central hub, the path is as follows: origin-hub A-central hub B-destination. At last, if hub A and hub B of the latter path are assigned to central hub C and central hub D respectively, the path is origin-hub A-central hub C-central hub D-hub B-destination. Note that demand nodes may be connected directly to a central hub.

It is worth mentioning that eliminating the first level of this network results in a classical hub network in which a path connecting a pair of nodes visits two hubs, at most. The hierarchical network considered in this paper requires more operations and therefore more coordination between different points of the network.

This paper considers such a network in which each node has a predetermined traffic capacity. It means that the total traffic which enters each node as a hub or central hub cannot be greater than its capacity. A fixed cost is charged if a demand node is selected to operate as a hub. Likewise, central hubs charge a fixed cost which is supposed to have a greater value. These costs depend on the node selected and its capacity. The number of hubs to be located and the maximum number of central hubs are given as well as the discount factors due to economies of scale. Also, delivery time restrictions are taken into account. In other words, the traffic demands of each node has to be delivered within a formerly specified time or a penalty cost will be charged depending on the lateness.

The number of the hubs and the maximum number of the central hubs are determined. The problem is to decide on the location of the hubs and central hubs in order to minimize the total costs. This problem is applicable in many real world problems, such as cargo delivery and data networks. For instance, in a cargo delivery system, different cities are considered as the nodes on the network so that the capacity of each node would be the maximum amount that may be transferred through that city. Therefore, cities with bigger airports are more likely to be chosen as hubs and central hubs, while cities that do not have any airport are unable to perform as a hub.

3. Model formulation

In this section, the mathematical model formulation of the problem is presented. Before presenting the model, notations used are described as follows.
Index of demand nodes \((i, m \in I)\);

Index of hub nodes \((j, j' \in H \subset I)\);

Index of central hub nodes \((l, v \in C \subset H)\);

Index of possible routes between node pairs \((z \in \{1, 2, 3\})\);

Number of hubs to be located;

Maximum number of central hubs to be located;

Discount rate of transportation costs between hubs and central hubs;

Discount rate of transportation costs between central hubs;

Unit quantity to be routed between demand nodes \(i\) and \(m\);

Unit transportation cost between nodes \(i\) and \(j\);

Traffic capacity of node \(j\) as a hub;

Traffic capacity of node \(j\) as a central hub;

Traffic capacity of the link between nodes \(i\) and \(j\);

Discount rate of transportation times between hubs and central hubs;

Discount rate of transportation times between central hubs;

The fixed cost charged if node \(i\) is selected as a hub;

The ratio between the fixed cost if node \(i\) is selected as a central-hub and a hub;

Time that the traffic at node \(i\) is ready to be released;

Transportation time between nodes \(i\) and \(j\);

The time that the traffic is delivered to node \(m\) from node \(i\) through path \(z\);

Delivery time limit for the destination node \(m\);

Unit penalty cost charged if the traffic quantity between demand nodes \(i\) and \(m\) is not delivered on time;

Binary variable. Equals 1 if demand node \(i\) is assigned to hub at node \(j\);

Binary variable. Equals 1 if hub at node \(j\) is assigned to central hub at node \(l\);

Thus, the model formulation of the problem is as follows.

\[
\begin{align*}
\text{minimize} & \quad \sum_{i} \sum_{m} q_{im} \left( \sum_{j} d_{ij} x_{ij} + \sum_{j} \sum_{l} \alpha_{hl} d_{jl} x_{ij} (1 - x_{mj}) y_{ij} + \sum_{l} \sum_{v} \alpha_{cv} d_{lv} \sum_{j} x_{ij} y_{ij} \sum_{j} x_{mj} y_{mj} + \right. \\
& \quad \left. \sum_{j} \sum_{v} \alpha_{hl} d_{hl} x_{jm} (1 - x_{mj}) y_{jv} + \sum_{f} d_{fm} x_{mf} \right) + \sum_{m} f_{m} \sum_{z} \max \{A_{imz} - \beta_{m}, 0\} \\
& \quad + h \sum_{i} x_{ii} + hr \times h_j \sum_{j} y_{jj} \\
\text{s.t.} & \quad \sum_{j} x_{ij} = P
\end{align*}
\]
The objective function (1) considers the total costs of the problem where the first term indicates total transportation costs on the network. The second term considers the penalty costs of lateness in delivery. The next two terms calculate the fixed costs of establishing hubs and central hubs respectively. Constraints (2) and (3) are implemented to take into account the number of hubs and the maximum number of central hubs respectively. Next constraint (4) considers the single assignment in the network. Equation (5) assures that a node is assigned to a hub if only the second node has been selected as a hub. The same is done by constraint (6) for assigning hubs to central hubs. Constraints (7) and (8) take the capacity limitations of hubs and central hubs into consideration, and equations (9) to (11) consider arc capacities, which are node-hub, hub-central hub and central hub-central hub arcs respectively. Constraints (12) to (14) are used to calculate the delivery times of different possible routes in order to determine the lateness times in the objective function (1).

4. Computational Results
In order to study the characteristics of the problem, the mathematical model described in Section 3 was
solved using GAMS 23.5 running on a PC (Intel® Core™ i7-2630QM CPU @2.00GHz 2.00 GHz, 6.00GB RAM, Windows 7 Ultimate).

For different sizes of the network, problems were generated randomly and solved with and without considering the capacity restrictions. In Table 1, the solutions of these two cases are compared.

<table>
<thead>
<tr>
<th>Number of: nodes, Hubs, Central hubs</th>
<th>Central hubs (capacitated)</th>
<th>Hubs (capacitated)</th>
<th>Central hubs (uncapacitated)</th>
<th>Hubs (uncapacitated)</th>
<th>Number of: nodes, Hubs, Central hubs</th>
</tr>
</thead>
<tbody>
<tr>
<td>8, 4, 2</td>
<td>4, 5</td>
<td>2, 7</td>
<td>4, 5</td>
<td>2, 7</td>
<td>8, 4, 2</td>
</tr>
<tr>
<td>2, 5, 10</td>
<td>2, 5</td>
<td>4, 7</td>
<td>1, 5, 10</td>
<td>4, 7</td>
<td>10, 5, 3</td>
</tr>
<tr>
<td>1, 3, 10, 12</td>
<td>5, 9, 11</td>
<td>1, 9, 12</td>
<td>3, 4, 10, 11</td>
<td>12, 7, 4</td>
<td></td>
</tr>
<tr>
<td>5, 6, 8, 10, 13</td>
<td>1, 2, 9, 15</td>
<td>5, 9, 10, 13</td>
<td>1, 6, 8, 12, 15</td>
<td>15, 9, 5</td>
<td></td>
</tr>
</tbody>
</table>

As can be seen, considering the capacity limitations in most cases has resulted in changes in the locations of the hubs and/or central hubs. For example, in the second problem, which has 10 nodes on the network, node 1 is selected as a central hub in the incapacitated problem, while in the capacitated problem this node is not even chosen as a hub. Such changes are also observed in other cases which indicates the role that capacities play in this problem. Not considering node and arc capacities, nodes without the required capabilities may be chosen as hubs or central hubs.

In order to evaluate the running times of the problem, different values were considered as the number of hubs and the maximum number of central hubs. In each case, 5 problems were generated at random and the average running times of these group are shown in Table 2.

<table>
<thead>
<tr>
<th>The average solution time</th>
<th>Max number of C- hubs</th>
<th>Number of hubs</th>
<th>Number of nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.874</td>
<td>2</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>12.237</td>
<td>3</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>38.121</td>
<td>2</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>107.512</td>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>794.357</td>
<td>5</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>2728.469</td>
<td>6</td>
<td>7</td>
<td>20</td>
</tr>
<tr>
<td>-</td>
<td>8</td>
<td>9</td>
<td>25</td>
</tr>
</tbody>
</table>

As can be seen, solving this problem in order to gain the optimal solutions is quite time consuming. When the number of the nodes on the network is greater than 10, no solution was obtained within an hour.

Figure 2 indicates the trend in the running times as the network size becomes bigger. It is observed that as the problem size increases, the running time is increased exponentially.

Figure 2. The trend of the running times as the network grows
Due to the complexity of the model, implementing heuristic and metaheuristic methods are suggested for this problem. A sensitivity analysis was performed to evaluate the effect of the node and arc capacities on the results. To do so, a problem with 10 nodes on the network was considered using the following value for its parameters. The distances between the nodes \( (d_{ij}) \) and the traffic demands \( (q_{im}) \) were given randomly in the following ranges respectively: \((100, 90)\) and \((10, 70)\). The range used for the fixed establishment costs of hubs is \((400, 10000)\) and \(h_{HR} \) equals to 0.6.

Figure 3 indicates the changes in the total costs as the capacities change. In this figure, the average value of the range used for the capacities are shown in the horizontal axis, while the vertical axis indicates the average values of the total costs of 5 problems with different capacities. The range length for the capacities was 200. For instance, value 300 on the horizontal axis represents the range \((200, 400)\).

As can be observed, the problem is quite sensitive to the capacities. However, increasing the capacities more than a specific value has no effect on the solution and the total costs. According to the results of the sensitivity analysis, calculation of the capacity level at which, having other parameters fixed, the minimum total cost is achieved could be considered as another problem. However, it should be reminded that in the present problem, nodes have different capacities and, in practice, it may be impossible to have an equal capacity in all the nodes of the network.

5. Conclusion

In this paper, a capacitated three-level hierarchical \( p \)-hub median problem has been considered where the problem is to decide on the nodes to be selected as hubs and those to choose as central hubs while minimizing the total costs. The number of hubs and the maximum number of central hubs are predetermined. The network at the top level (central hubs) is complete and two next levels are star networks. A mixed integer non-linear formulation for the problem is given where arcs and nodes of the network have limited traffic capacities and there are restrictions over the delivery time of the traffics.

The mathematical model formulation of the problem were presented and solved using GAMS. Results indicated that the limitations considered in this problem, which has not formerly been studied in the literature, has a considerable effect on the solution of the problem. The running times have shown that trying to gain the optimal solutions using the mathematical model will be notably time-consuming and therefore implementing heuristic and meta-heuristic methods are suggested for this problem. The sensitivity analysis indicated that changes in node capacities up to a specific number will change the results considerably, however, increasing the capacities will not affect the results when that certain point is reached. Thus, determining this point may be considered as another problem.
References


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