On the bounds and norms of a particular Hadamard Exponential Matrix

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Abstract:

In this note we study a particular $n \times n$ Hadamard exponential Matrix of the form $e^{H_n} = \left[ e^{h_n} \right]_{i,j}$ where $h_n = a + \max(i, j) - 1$ and $a$ is a fixed real number. We find $l_p$ norm and two upper bounds of the spectral norms of this matrix. Finally we represent some properties of Hadamard inverse, its product and determinant of the inverse of this matrix.

Keywords: Hadamard exponential matrix, upper bounds, $l_p$ norm, inverse.

Introduction:

Some authors studied the matrix $A = [a_{ij}]$ which entries depend on $i, j$. In [6] Solak and Bahsi studied the matrix of the form $B = [b_{ij}]$ where $b_{ij} = a + \max(i, j) - 1$. They studied some properties of this matrix. Bozkort in [3] computed the spectral norms of the matrices related with integer sequences. In [2] Solak and Bozkort determined bounds for the spectral and $l_p$ norm of Cauchy-Hankel matrices of the form $H_n = \left[ \frac{1}{g + k h} \right]_{i,j=0}^n$ where number $k$ is defined by $i + j = k$ and $g, h$ are any positive numbers. Civciv and Turkmen in [4] established a lower bound and upper bound for the $l_p$ norms of the Khatri-Rao product of Cauchy-Hankel matrix of the form $H_n = \left[ \frac{1}{2 + (i + j)} \right]_{i,j=0}^n$.

In this note we study a particular $n \times n$ Hadamard Exponential Matrix of the form $e^{H_n} = \left[ e^{a+\max(i,j)-1} \right]_{i,j}$, in exact of the form

$$
\begin{bmatrix}
  e^a & e^{a+1} & \cdots & e^{a+n-1} \\
  e^{a+1} & e^{a+1} & \cdots & e^{a+n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  e^{a+n-1} & e^{a+n-1} & \cdots & e^{a+n-1}
\end{bmatrix}
$$

(1)

We find $l_p$ norm, two upper bounds of the spectral norms of this matrix. Finally we represent an application to Hadamard inverse, its product and inverse of this matrix. All definitions and statements of this section are available in references [1-5].

1. Preliminary Definitions
Let \( A = [a_{ij}] \) is an \( n \times n \) matrix, then we define the maximum column length norm \( C_1(.) \) and maximum row length norm \( r_1(.) \) of matrix \( A \) by

\[
C_1(A) = \max_j \sqrt{\sum_i |a_{ij}|^2}, \quad r_1(A) = \max_i \sqrt{\sum_j |a_{ij}|^2} .
\] (2)

Hadamard exponential and Hadamard inverse of this matrix is defined by \( e^{sA} = e^{sa} \) and \( A^{-1}_\circ = \left( \frac{1}{a_{ij}} \right) \). The \( l_p \) norm of \( A \) is defined by

\[
\|A\|_p = \left( \sum_{i,j} |a_{ij}|^p \right)^{1/p}. \tag{3}
\]

For \( p = 2 \) this norm is called Frobenius or Euclidean norm and showed by \( \|A\|_F \). The spectral norm of \( A \) is defined by

\[
\|A\|_2 = \max_{1 \leq i \leq n} |\lambda_i|, \tag{4}
\]

where \( \lambda_i \) are the eigenvalues of the matrix \( AA^H \). Also \( A^H \) is conjugate transpose of the matrix \( A \). The inequality between the Frobenius and spectral norm is as follows

\[
\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F \cdot \tag{5}
\]

Let \( A = [a_{ij}] \), \( B = [b_{ij}] \) and \( C = [c_{ij}] \) are \( m \times n \) matrices. Then Hadamard product of \( A \) and \( B \) is defined by \( A \circ B = [a_{ij}b_{ij}] \). If \( A = B \circ C \) then we have

\[
\|A\|_2 \leq r_1(B) c_1(C). \tag{6}
\]

It is known that

\[
\sum_{k=0}^{n-1} x^k = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}. \tag{7}
\]

2. Main Results

**Theorem 2.1.** Let \( e^{\lambda A} \) be as in (1) then the \( l_p \) norm of \( e^{\lambda A} \) is;

\[
\|e^{\lambda A}\|_p = e^\lambda \left[ \frac{e^{p(n+2)} - e^{p(n+1)} + e^p - 1}{(e^p - 1)^2} \right]^{1/2}. \tag{8}
\]

**Proof.** By definition of \( e^{\lambda A} \), we have

\[
e^{\lambda A} = \begin{bmatrix}
e^0 & e^1 & \cdots & e^{n-1} \\
e^1 & e^1 & \cdots & e^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
e^{n-1} & e^{n-1} & \cdots & e^{n-1}
\end{bmatrix}.
\]

So we have

\[
\|e^{\lambda A}\|_p = e^\lambda \left( 1 + e^p + e^{2p} + e^{3p} + \cdots + e^{(n-1)p} \right) + (e^p + e^p + e^{2p} + \cdots + e^{(n-1)p}) + \cdots + (e^{(n-1)p} + \cdots + e^{(n-1)p}) = e^\lambda \sum_{k=1}^{n} (2k - 1)e^{(k-1)p}
\]
By using equation (7) we can write
\[
\|\mathbf{e}^{H_k}\|_p = e^{(a-1)p}\left( 2ne^{p(n+2)} - (n+1)e^{p(n+1)} + e^p - e^{p(n+1)} - 1 \right) e^{p(n+1)} - 1 } )\frac{1}{(e^p - 1)^2}.
\]
Thus we have
\[
\|\mathbf{e}^{H_k}\|_p = e^{(a-1)p}\left( 2ne^{p(n+2)} - (n+1)e^{p(n+1)} + e^p - e^{p(n+1)} - 1 \right)\frac{1}{(e^p - 1)^2}.
\]

By taking \(\frac{1}{p}\) th power from the both side of the above equality we get
\[
\|\mathbf{e}^{H_k}\|_p = e^{(a-1)p}\left( \frac{2n(e^{p(n+2)} - (n+1)e^{p(n+1)} + e^p - e^{p(n+1)} - 1)\frac{1}{(e^p - 1)^2}}{p} \right).
\]

**Corollary 2.2.** If we put \(p = 2\) in (8) then we get the Euclidean norm of \(\mathbf{e}^{H_k}\), that is equals
\[
\|\mathbf{e}^{H_k}\|_2 = e^{(a-1)p}\left( \frac{2n(e^{p(n+2)} - (n+1)e^{p(n+1)} + e^p - e^{p(n+1)} - 1)\frac{1}{(e^p - 1)^2}}{2} \right).
\]

**Theorem 2.3.** Let \(\mathbf{e}^{H_k}\) be as in (1) then we have
\[
\|\mathbf{e}^{H_k}\|_2 = e^{(a-1)p}\left( \frac{2n(e^{p(n+2)} - (n+1)e^{p(n+1)} + e^p - e^{p(n+1)} - 1)\frac{1}{(e^p - 1)^2}}{2} \right) \leq \|\mathbf{e}^{H_k}\|_2 \leq e^{(a-1)p}\left( \frac{2n(e^{p(n+2)} - (n+1)e^{p(n+1)} + e^p - e^{p(n+1)} - 1)\frac{1}{(e^p - 1)^2}}{2} \right).
\]

**Proof.** It follows from corollary 3.2 and relation (5).

**Theorem 2.4.** Let \(\mathbf{e}^{H_k}\) be as in (1) then we have
\[
\|\mathbf{e}^{H_k}\|_2 \leq (ne^{2(a+n-1)})^{\frac{1}{2}} \left( (n-1)e^{2(n+1)} + 1 \right)^{\frac{1}{2}}.
\]

**Proof.** By definition of \(\mathbf{e}^{H_k}\) we have
\[
\left[ e^a e^{a+1} \cdots e^{a+n-1} \\
e^{a+1} e^a \cdots e^{a+n-1} \\
\vdots \vdots \vdots \\
e^{a+n-1} e^{a+1} \cdots e^{a+n-1} \right] = \left[ e^a e^{a+1} \cdots 1 \\
e^{a+1} e^a \cdots 1 \\
\vdots \vdots \vdots \\
e^{a+n-1} e^{a+1} \cdots 1 \right] \left[ 1 1 \cdots e^{a+n-1} \\
1 1 \cdots e^{a+n-1} \\
\vdots \vdots \vdots \\
1 1 \cdots 1 \right] = A \circ B.
\]

By definition of row maximum length norm and column maximum length norm we have
\[
r_i(A) = \max_i \sqrt{\sum_j |A_{ij}|^2} = \sqrt{ne^{2(a+n-1)}} = \sqrt{ne^{2(a+n-1)}}, \quad C_j(B) = \max_j \sqrt{\sum_i |B_{ij}|^2} = \sqrt{(n-1)e^{2(n+1)} + 1}.
\]

According to (6) we know \(\|\mathbf{e}^{H_k}\|_2 \leq r_i(A)c_j(B).\) Thus we have
\[
\|\mathbf{e}^{H_k}\|_2 \leq (ne^{2(a+n-1)})^{\frac{1}{2}} \left( (n-1)e^{2(n+1)} + 1 \right)^{\frac{1}{2}}.
\]

**Theorem 2.5.** Let \(\mathbf{e}^{H_k}\) be as in (1) then
\[
\det(\mathbf{e}^{H_k}) = (1-e)^{n-1} e^{(n-1)(n+1)}.
\]

**Proof.** By definition of \(\mathbf{e}^{H_k}\) we can write
\[
\det(e^{H_n}) = (e^n)^n = \begin{bmatrix}
 e^0 & e^1 & \cdots & e^{n-1} \\
 e^1 & e^1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 e^{n-1} & e^{n-1} & \cdots & e^{n-1}
\end{bmatrix}.
\]

By using elementary row operations we get
\[
\det(e^{H_n}) = (e^n)^n = (1-e)^{n-1}e^{n-1}e^{-\frac{n(n-1)}{2}} = (1-e)^{n-1}e^{\frac{n(n-1)}{2}}.
\]

We need a lemma from matrix algebra to proving next theorem.

**Lemma 2.6.** Let \( A \) is an \( n \times n \) nonsingular matrix, \( b \) is an \( n \times 1 \) matrix and \( c \) is a real number. If we take \( M = \begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \) then we have \( M^{-1} = \begin{bmatrix} A^{-1} + \frac{1}{k}A^{-1}bb^TA^{-1} - \frac{1}{k}A^{-1}b \\ -\frac{1}{k}b^TA^{-1} & \frac{1}{k} \end{bmatrix} \), where \( k = c - b^TA^{-1}b \).

**Proof.** By multiplying of two above mentioned matrices we have
\[
\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} A^{-1} + \frac{1}{k}A^{-1}bb^TA^{-1} - \frac{1}{k}A^{-1}b \\ -\frac{1}{k}b^TA^{-1} & \frac{1}{k} \end{bmatrix} = I_{n+1}.
\]

Thus the proof is completed.

**Theorem 2.7.** Let \( e^{H_n} \) be as in (1) then the inverse of \( e^{H_n} \) is a tridiagonal matrix, that is
\[
(e^{H_n})^{-1} = e^{-A} = \begin{bmatrix}
 \frac{1}{e-1} & \frac{1}{e-1} & 0 & 0 & 0 & \cdots & 0 \\
 \frac{1}{e-1} & \frac{-e+1}{e(e-1)} & \frac{1}{e(e-1)} & 0 & 0 & \cdots & 0 \\
 0 & \frac{1}{e(e-1)} & \frac{-e+1}{e^2(e-1)} & \frac{1}{e^2(e-1)} & 0 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
 0 & 0 & \frac{1}{e^2(e-1)} & \frac{1}{e^3(e-1)} & \frac{-e+1}{e^3(e-1)} & \frac{1}{e^3(e-1)} & 0 \\
 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}.
\]

**Proof.** We prove this theorem by principal mathematical induction on \( n \).

The result is true for \( n = 2 \) that is
If \( e^{H_n} = e^a \begin{bmatrix} e^0 & e^1 \\ e^1 & e^1 \end{bmatrix} \) then we have \( (e^{H_n})^{-1} = e^{-a} \begin{bmatrix} -1 & 1 \\ e^{-1} & e^{-1} \\ 1 & -1 \\ e^{-1} & e(e-1) \end{bmatrix} \).

Now assume that it is true for \( n - 1 \), that is

\[
(e^{H_n})^{-1} - e^{-a} = \begin{bmatrix}
-1 & 1 \\
-1 & e^{-1} \\
1 & -1 \\
e^{-1} & e(e-1) \\
\end{bmatrix}
\]

By assumptions of theorem we can take

\[
e^{aH_{n-1}} = e^a \begin{bmatrix} e^0 & e^1 & \cdots & e^{n-2} \\ e^1 & e^1 & \cdots & e^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ e^{n-1} & e^{n-1} & \cdots & e^{n-2} \end{bmatrix}, \quad b = \begin{bmatrix} e^{a+n-1} \\ e^{a+n-1} \\ \vdots \\ e^{a+n-1} \end{bmatrix}, \quad b^T = \begin{bmatrix} e^{a+n-1} & e^{a+n-1} & \cdots & e^{a+n-1} \end{bmatrix}, \quad A^{-1} = e^{H_{n-1}}.
\]

So we get

\[
A^{-1}b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e \end{bmatrix}, \quad b^T A^{-1} = \begin{bmatrix} 0 & 0 & \cdots & e \end{bmatrix},
\]

\[
k = e^{a+n-1} - b^T A^{-1} b = e^{a+n-1} - \begin{bmatrix} 0 & 0 & \cdots & e \end{bmatrix} e^{a+n-1} = e^{a+n-1} - e^{a+n} = e^{a+n-1} (1-e) = -e^{a+n-1} (e-1).
\]

So we have

\[
\frac{1}{k} = -\frac{1}{e^{a+n-1} (e-1)}.
\]

According to the previous lemma we have

\[
(e^{H_n})^{-1} = \begin{bmatrix} A^{-1} + \frac{1}{k} A^{-1} b b^T A^{-1} & -\frac{1}{k} A^{-1} b \\ -\frac{1}{k} b^T A^{-1} & 1 \end{bmatrix}.
\]

So by substituting values of \( A, b \) and \( k \) we get the result.
Theorem 2.8. Determinant of \((e^{H_n})^{-1}\) is;

\[
\det((e^{H_n})^{-1}) = (1 - e)^{-n}e^{(-na + n(n-1)/2)}.\]

Proof. As we mentioned in theorem 3.5, we have

\[
\det(e^{H_n}) = (1 - e)^{n-1}e^{(na - n(n-1)/2)}.\]

So we can write

\[
\det((e^{H_n})^{-1}) = \frac{1}{\det(e^{H_n})} = \frac{1}{(1 - e)^{n-1}e^{(na - n(n-1)/2)}} = (1 - e)^{-n}e^{(-na + n(n-1)/2)}.\]

Theorem 2.9. Let \(e^{H_n}\) be as in (1), then the determinant of Hadamard inverse of \(e^{H_n}\) equals

\[
\det((e^{H_n})^{(-1)}) = e^{-na}(e - 1)^{n-1}e^{(n-1)x(n+2)}.\]

Proof. By definition of \((e^{H_n})^{(-1)}\) we have

\[
det \left( (e^{H_n})^{(-1)} \right) = det \left[ \begin{array}{ccc} -a & (-a+1) & -(a+n-1) \\ e & e & \cdots e \\ -(a+1) & -(a+1) & -(a+n-1) \\ e & e & \cdots e \\ \vdots & \vdots & \vdots \\ -(a+n-1) & -(a+n-1) & -(a+n) \\ e & e & \cdots e \end{array} \right] = e^{-na} \left[ \begin{array}{ccc} 1 & e^{-1} & \cdots e^{-(n-1)} \\ e^{-1} & e^{-1} & \cdots e^{-(n-1)} \\ \vdots & \vdots & \vdots \\ e^{-(n-1)} & e^{-(n-1)} & \cdots e^{-(n-1)} \end{array} \right].\]

By using elementary row operations we get

\[
det((e^{H_n})^{(-1)}) = e^{-na} \left[ \begin{array}{ccc} 1 & e^{-1} & \cdots e^{-(n-1)} \\ e^{-1} & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e^{-(n-1)} & e^{-(n-1)} & \cdots & 0 \end{array} \right].\]

So by expanding this determinant we have

\[
\det(e^{H_n})^{(-1)} = e^{-na}(e - 1)^{n-1}e^{(n-1)(n+2)}.\]

Theorem 2.10. The \(l_p\) norm of \((e^{H_n})^{(-1)}\) is

\[
\| (e^{H_n})^{(-1)} \|_p = e^{(1-p)} \left( \frac{(2n-1)e^{-p(n+2)} - (2n+1)e^{p(n+1)} + e^{-p} - 1}{(e^{p} - 1)^2} \right)^{1/p}.\]

Proof. The proof is similar to the theorem (2.1).
Theorem 2.11. The inverse of matrix \((e^{nH_n})^{(-1)}\) is a tridiagonal matrix like as follows.

\[
\begin{bmatrix}
\frac{e^0}{e-1} & \frac{e}{e-1} & 0 & 0 & 0 & \cdots & 0 \\
\frac{e}{e-1} & \frac{e^2}{(e-1)} & \frac{e^{(e+1)}}{(e-1)(e^2)} & 0 & \cdots & 0 \\
0 & \frac{e^2}{(e-1)} & \frac{e^3}{(e-1)(e^2)} & \frac{e^4}{(e-1)(e^3)} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & \frac{e^n}{(e-1)} & \frac{e^{n-1}}{(e-1)} \\
\end{bmatrix}.
\]

Proof. We can prove this theorem by principal mathematical induction on \(n\) similar to theorem (3.7).

References